The Mathematics of Psychotherapy: A Nonlinear Model of Change Dynamics

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Abstract. *Psychotherapy is a dynamic process produced by a complex system of interacting variables. Even though there are qualitative models of such systems the link between structure and function, between network and network dynamics is still missing. The aim of this study is to realize these links. The proposed model is composed of five state variables (P: problem severity, S: success and therapeutic progress, M: motivation to change, E: emotions, I: insight and new perspectives) interconnected by 16 functions. The shape of each function is modified by four parameters (a: capability to form a trustful working alliance, c: mentalization and emotion regulation, r: behavioral resources and skills, m: self-efficacy and reward expectation). Psychologically, the parameters play the role of competencies or traits, which translate into the concept of control parameters in synergetics. The qualitative model was transferred into five coupled, deterministic, nonlinear difference equations generating the dynamics of each variable as a function of other variables. The mathematical model is able to reproduce important features of psychotherapy processes. Examples of parameter-dependent bifurcation diagrams are given. Beyond the illustrated similarities between simulated and empirical dynamics, the model has to be further developed, systematically tested by simulated experiments, and compared to empirical data.*

Key Words: Mathematical modeling, psychotherapy, process research, common factors, computer simulation, nonlinear dynamics

INTRODUCTION: THE COMMON FACTORS APPROACH

During the last decades, efforts were made to identify the factors contributing to the outcome and explaining the outcome variance of psychotherapy. Whereas the main focus of European and American psychotherapy research was on specific or technical factors, process-outcome research and integrative approaches were more interested in the common factors shared by all

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psychotherapy schools. Specific factors are related to the therapeutic actions which – in correspondence to the etiological model of a disorder – create the therapeutic effect (Wampold, 2010). Given this implicit linear causality and linking the therapeutic action to the outcome, the emphasis then lies on the correct realization of treatment techniques (e.g., by the adherence to a manual). In consequence, there is no real need for understanding the complex interaction of non-technique-related factors.

The common factors approach proposes that different schools and evidence-based practices in psychotherapy share factors that account for the bigger part of the effectiveness of psychological treatments (Carr, 2008; Imel & Wampold, 2008; McAleavey & Costonguay, 2015; Wampold & Imel, 2015). Common factors are unspecific in the sense that they are not specific to a certain therapy approach or treatment technique. Amongst others, the common factors model emphasizes the cooperation and interaction of client and therapist as well as their personalities, and it focuses more on the structure of the therapeutic work and the way it is transported or made plausible to the client than on the specific contents and techniques of the approach (Wampold, 2010). Reviews of thousands of studies identified a great number of unspecific factors concerning the client (Bohart & Tallman, 2010; Clarkin & Levy, 2004; Duncan, Miller, & Sparks, 2004; Garfield, 1994), the therapist (Beutler, Malik, Alimohamed, Harwoood, Talebi, et al., 2004), the client-therapist interaction and working alliance (Crits-Christoph, Gibbons, Hamilton, Ring-Kurtz, & Gallop, 2011; Horvath & Symonds, 1991; Martin, Garske, & Davis, 2000; Norcross, 2002; Norcross & Lambert, 2005), and other aspects like empathy (Bohart, Eliott, Greenberg, & Watson, 2002), expectations, rituals, or treatment rationale (Duncan, Miller, Wampold, & Hubble, 2010; Grencavage & Norcross, 1990; Sparks & Duncan, 2010).

Some research groups are engaged in narrative or statistical classifications of the diversity of common factors (Tschacher, Junghan, & Pfamatter, 2014). One of the first were Grencavage and Norcross (1990) who reviewed accounts of common factors in 50 publications, with 89 factors in all, from which they selected the 35 most common ones and grouped them into five areas: client characteristics, therapist qualities, change processes, treatment structure, and therapeutic relationship. A review of common factors research (Imel & Wampold, 2008) suggested that 30% to 70% of the variance in therapy outcome was due to common factors. A summary of research by Laska, Gurman and Wampold (2014) suggested that 11.5% of variance in therapy outcome could be explained by goal consensus or collaboration, 9% by empathy, 7.5% by therapeutic alliance, 6.3% by positive regard or affirmation, 5.7% by congruence/genuineness, and 5% by therapist factors; in contrast, treatment method accounted for only 1% of outcome variance.

Psychotherapy research should not only try to find statistical evidence that certain factors contribute to successful outcomes; it must also be able to fomulate explanations for how and why those factors contribute to effects, that is, the mechanisms through which successful psychotherapy leads to change

(Kazdin, 2005, 2009). One way to understand these mechanisms is to explicate the interrelations of the factors. The "generic model of psychotherapy" designed by Orlinsky and Howard (1986, 1987, later elaborations by Orlinsky, Grawe, & Parks, 1994; Orlinsky, Ronnestad, & Willutzky, 2004) was the first model that tried to integrate clusters of variables taken from a systematic review on process-outcome studies by a graphical network model. Other graphical models followed (e.g., Grawe, 1995; Schiepek, Eckert, Aas, Wallot, & Wallot, 2015) and some authors noted that the "… client variables are in a dynamic and ever changing context of therapist variables and behaviour" (Clarkin & Levy, 2004, p. 215). Also there are empirical (bottom-up) models, that make use of connectivity analyses of time series data like vector autoregression models or state space models (Tschacher, Baur, & Grawe, 2000). However, there appear to be only scarce attempts to theorize the interconnectedness and dynamics of therapeutic variables by means of formal and straightforward a top-down models. This may be astonishing since it seems evident that if the explanandum is change dynamics the explanans must have the structure of a system which in some way is able to produce dynamics.

COMPUTER SIMULATIONS IN CLINICAL PSYCHOLOGY

Without concern to psychotherapy, simulation models of dynamic mental diseases were proposed more than two decades ago. In one account, the dynamics of depression were simulated by a set of production rules (if-then-rules), interconnecting eight variables taken from the idiographic system model of a client (Schaub & Schiepek, 1992). Another simulation study was based on coupled nonlinear difference equations representing five crucial factors (cognitive basic disorders, expressed emotions, stress, withdrawal, delusions) determining the long-term evolutionary patterns of schizophrenia. Depending on the control parameters of the model, the most prominent patterns described by Ciompi and Mueller (1976) could be reproduced (Schiepek, Schoppek, & Tretter, 1992). Kriz (1992) modelled conflictual family dynamics by a set of equations taken from population dynamics. A recent study (Demic & Cheng, 2014) reproduced different disease states of depression (depressive episode, recovery, relapse, remission) by a noise-driven dynamic systems model of one state variable.

Following a mathematical model of marriage dynamics (Gottman, Murray, Swanson, Tyson, & Swanson, 2002), the Gottman group modeled the co-evolution of emotional valences expressed by a therapist and his client (Liebovitch, Peluso, Norman, Su, & Gottman, 2011; Peluso, Liebovitch, Gottman, Norman, & Su, 2012). Two differential equations represent the emotional valence of the client and the therapist (respectively) with parameters representing the client's and therapist's inertia to change, their emotional states independent of each other, and the coupling or reactivity strengths. The influence functions of the client to the therapist and vice versa consist of three segments of linear functions each defining the gradient of emotional changes

which the client exerts on the therapist and the therapist exerts on the client. These segments of linear gradients over the definition range of the system variables (client valence, therapist valence) limit the model. In consequence, it leads to the prediction of stable fixed-point attractors of the therapeutic relationship at the intercept of the valence functions, or to drop-outs, depending on the initial conditions in the two-dimensional phase portrait. Crisis-repair sequences (e.g., Gumz, Brähler, Geyer, & Erices, 2012) and other dynamic phenomena of the therapeutic process are not in the scope of the model. This approach is an important step towards the formalization and simulation of psychotherapy dynamics, despite its limitations in form of the segmented linear functions and the narrow scope on dyadic relationship valences.

MODEL CRITERIA AND REQUIREMENTS

Models are different from reality. By definition, they are selective, take a certain perspective on the phenomena under consideration, and abstract from many details. In consequence, models reduce complexity. At the other side, models should be in some way realistic and fulfill some plausibility criteria. A good model should be capable to plausibly reproduce or at least not contradict the given knowledge on an empirical system, and have the surplus of explaning beyond the assumptions made in the model or in the mere verbal description of a system. One source of this added value are emergent qualities like self-organized thresholds, synchronization patterns, phase transitions, or chaotic dynamics, which are core concepts of synergetics. Synergetics models the emergence of patterns that rise from the interaction of the componets of a non-linear system, depending on the parameters (modulating the interaction of the systems components) and on the boundary conditions of the system (for a brief outline of the synergetic program see Schiepek et al., 2015; a more comprehensive exposition is given in Haken, 2004; Haken & Schiepek, 2010). Specifically, a model on psychotherapy dynamics should fulfill the following criteria:

1. Empirical findings. The model should reflect empirical findings on the course and outcome of psychotherapy. For example, it should respect the significant role of client factors like motivation to change, cognitive and behavioral skills, competencies (e.g., in mentalization and emotion regulation), and other resources (Bohart & Tallman, 2010; Clarkin & Levy, 2004). Symptom severity, therapeutic progress, and motivation to change should be related to each other. The model should clarify by which state variables therapeutic actions could have an impact on the dynamics of the client. The role of the client-therapist-interaction or working alliance (Crits-Christoph, Gibbons, Hamilton, Ring-Kurtz, & Gallop, 2011) should be reflected, and the interconnectedness of emotions, insight, and confrontation with traumata or inner conflicts should be respected. The role of problem actualization (activation of negative schemata) and the necessity of a clarification perspective on problems – both stressed by Grawe's (1995, 2004) psychological concept of psychotherapy – should be represented. Psychodynamic, cognitive-behavioral, and resource-oriented contributions to understanding psychotherapy should be linked.

2. The dynamics of change processes. The dynamics of a client should remain unchanged under unchanged conditions, i.e., if neither variations in the trait parameters of the client nor any input from the therapist or from other social environments (at least noise) are introduced. However, in some cases (i.e., given specific parameter values) spontaneous recovery or intermittent phases of recovery or deterioration should be possible without interventions and external forces. The simulated dynamics have to be chaotic, since findings suggest that psychotherapy shows features of deterministic chaos as irregularity and dynamic complexity, restricted predictability, or sensitive dependency of the trajectories on initial conditions or on minor inputs or fluctuations (Kowalik, Schiepek, Kumpf, Roberts, & Elbert, 1997; Schiepek et al., 1997; Schiepek, Heinzel, Karch, Aas, & Strunk, 2014; Strunk & Schiepek, 2006). The theory of self-organizing systems (Haken, 1990, 2004; Haken & Schiepek, 2010; Mahoney, 1991; Pincus, 2009) defines a specific dependency of pattern or phase transitions on control parameters: continuous changes of parameter values trigger discontinuous pattern transitions. Psychotherapy dynamics are characterized by nonstationarities and discontinuous transitions (e.g., Haken & Schiepek, 2010; Hayes, Feldman et al., 2007; Hayes, Laurenceau, Feldman, Strauss, & Cardaciotto, 2007; Heinzel, Tominschek, & Schiepek, 2014; Miller & C'de Baca, 2001; Schiepek et al., 2014) reflecting sudden gains or sudden losses (e.g., Stiles, Leach, Barkham, Lucock, Iveson, et al., 2003). The theory requires a change of parameters to produce therapeutic phase transitions, whereas a time-limited displacement of the dynamics without a shift of parameters should be resorbed by the dominating attractor – a restabilization which prevents any long-term consequences on the therapeutic outcome. Generally, a nonlinear dynamic systems model of psychotherapy should correspond to the generic principles of therapeutic self-organization in human systems (Schiepek et al., 2015).

3. Functions. The functions of the model connect the respective variables, and therby represent psychological mechanisms of their phenomenological relationship. Existing empirical psychological knowledge forms the basis for modelling these connections and is drawn from such different areas as emotion, motivation, cognitive, or social psychology, psychotherapy research, and clinical experience. Furthemore, the shape of these functions is susceptible to individual differences and thus modified by clients dispositions and traits. It is here, where personal resources, competencies, skills, or expectations enter the theoretical model.

4. Model testing and empirical foundation. Models should be testable. One way of doing this is by manipulation of the simulation runs (simulated experiments). Variables can be set at different initial conditions or varied at certain iteration steps, parameters can be changed, and noise can be added. Then we observe how the system evolves. A minor criterion for the quality of models is the stability of the simulation-runs over a certain number of iterations. During the simulation period, the system should not collapse due to an infinite increase or decrease of the variables. Ideally, the simulated dynamics will be compared to

empirical dynamics given by time series representing the constructs or variables of the model (see the methodology of complexity science described in Haken & Schiepek, 2010; Strunk & Schiepek, 2006). If available, initial conditions can be taken and parameters can be estimated from empirical time series to optimize the fit between the empirical and the simulated dynamics. A successful optimization of simulations by the use of empirical data can be taken as an empirical support of the model.

THE MODEL

The model proposed here focuses on the cognitive-emotional dynamics of the client. Consequently we respect that the client is at the focus of the therapeutic engagement, contributes most to the outcome, and is the system where the therapeutic self-organization takes place. The model contains five variables (in terms of synergetics: order parameters), four control parameters, and 16 functions relating the variables to each other or creating autocatalytic effects within some variables.

The Variables

The variables of the model can be seen as psychological states with varying intensities on a given time scale. Here we suppose a sampling rate of once per day, i.e., each iteration of the simulation can be interpreted as a daily measurement of the variables. Such kind of assessment is realized by the Therapy Process Questionnaire (TBQ, see below Table 1). The model and interrelations of the variables are exposed in Figures 1 and 2.

The following variables constitute the model: (P) problem intensity, symptom severity, experienced conflicts or incongruencies; (S) success, therapeutic progress, goal attainment, confidence in a successful therapy course; (M) motivation to change, readiness for the engagement in therapy-related activities and experiences; and (E) Emotions. This is a bidimensional variable representing dysphoric emotions at one end of its dimension (e.g., composed by anxiety, grief, shame, feelings of guilt, anger, rage) and positive mental experiences at the other end (e.g., joy, self-esteem, flow). This definition of polarity is due to the factor analysis of the TPQ, by which the empirical reference time series were produced (Haken & Schiepek, 2010; Honermann, 2001). The last variable is (I) Insight, getting new perspectives on personal problems, motivations, cognitions, or behaviors (clarification perspective in the sense of Grawe), confrontation with conflicts, avoided behaviors and cognitions, or with repressed material like traumatic experiences

The term "emotion" as it is used here corresponds to the integration of affect with memories of previous affective and cognitive experiences and with current somatic sensations (Damasio, 1994). In contrast to "affect" as a sensorial experience in response to internal or external stimuli expressed with physiological and motor responses, "emotion" is a complex set of affects with mental

representations generated in association with previous memories and bodily experiences (Renaud & Zacchia, 2012).

In particular, only motivation to change (M) and insight (I) are common factors in a narrow sense. Emotions usually are not subsumed under common factors (Duncan et al., 2010; Tschacher et al., 2014), and problem intensity (P) and success (S) may be seen as momentary micro-outcomes of psychotherapy (Orlinsky et al., 1994, 2004).

The emphasis on client-related and treatment-unspecific factors does not imply that treatment-related factors are not taken into account. These factors can be seen as the interventions onto the dynamics of the system, which can trigger each of the variables of the model in a different way and affect the dynamics at different times with different impact (e.g., depending on the stability or instability of the dynamics). By this, treatment-specific factors are not part of the model, but enter the stage as an input from the outside.

The Parameters

Parameters are quantities mediating the interactions between variables. Depending on their values the effect of one variable on another can be intensified or reduced, activated or inhibited. Formally they modify the function defined for the relationship of two (or more) variables to each other. Psychologically, parameters can be interpreted as trait variables or dispositions changing at a slower time scale than the variables or states of a system. In terms of synergetics, the change of system parameters drives phase transitions of the dynamics.

The model includes four parameters: (*a*) *Working alliance* is the capability to enter a trustful attachment and cooperation with the therapist, quality of the therapeutic relationship, and interpersonal trust. On the one hand, this parameter signifies the disposition to engage in a trustful relationship (attach-ment disposition) and at the other hand it refers to the realized quality of the therapeutic bond. (*c*) *Cognitive competencies* are capacities for mentalization and emotion regulation, mental skills in self-reflection, or structure level in the sense of the Operationalized Psychodynamic Diagnostics (OPD, www.opd-online.net). (*r*) *Behavioral resources and skills* are those that can be applied to problem solving. (*m*) Motivation to change is a trait that encompasses the degree of learned self-efficacy, positive expectations in one's development, reward expectation, or "health plan" in the sense of the control mastery theory (Silberschatz, 2009; Weiss, 1993).

Table 1 clarifies the correspondence of the five variables and of parameter *a* (working alliance, trustful relationship to the therapist) to the factors (subscales) of the Therapy Process Questionnaire (TPQ). For details of the factor analysis see Haken and Schiepek (2010) and Honermann (2001). An example of two items of each factor may illustrate how the variables represent the respective cognitive-emotional dynamics of the client. Parameters *c*, *r*, and *m* are not represented in the questionnaire. In clinical routine applications, the TPQ is administered by the Synergetic Navigation System (an internet-based real-

Table 1. The Correspondence of the 7 Factors (Subscales) of the TPQ to the Variables of the Model and to Parameter *a*.

| Model variables	TPQ factors (subscales)
(P) Problem intensity, symptom severity, experienced conflicts or incongruencies	Complaints and problem pressure (item examples: Intensity of problems [visual analog scale]; Today the symptoms impaired my daily routines)
(S) Success, therapeutic progress, goal attainment, confidence in a successful therapy course	Therapeutic progress and confidence in a successful development (item examples: Today I felt hopeful to achieve my therapeutic goals; Today I felt able to cope with situations in which I felt helpless up to now)
(M) Motivation to change, readiness for the engagement in therapy-related activities and experiences	Motivation to change, engagement in therapy-related activities (item examples: Today I was engaged in solving my problems; Today I adressed myself to problems I avoided thus far)
(E) Emotions (bidimensional variable representing dysphoric emotions at the positive pole and positive mental experiences at the negative pole)	Dysphoric emotions (item examples: Today I felt anxiety; Today I felt joy)
(I) Insight, new perspectives on personal problems, cognitions, or behaviors, confrontation with conflicts or avoided behaviors/cognitions	Opening of perspectives, personal innovations (item examples: Today I developed new perspectives on my life; Today I got new ideas for solving my problems)
Model parameter	
(*a*) Working alliance, capability to enter as well as experience of a trustful attachment and cooperation with the therapist	Relationship quality and trust in therapists (item examples: Today, the cooperation with the therapist was helpful and supporting; Today, the relationship to my therapist was trustful)
No correspondence	Ward atmosphere and relationship with fellow patients
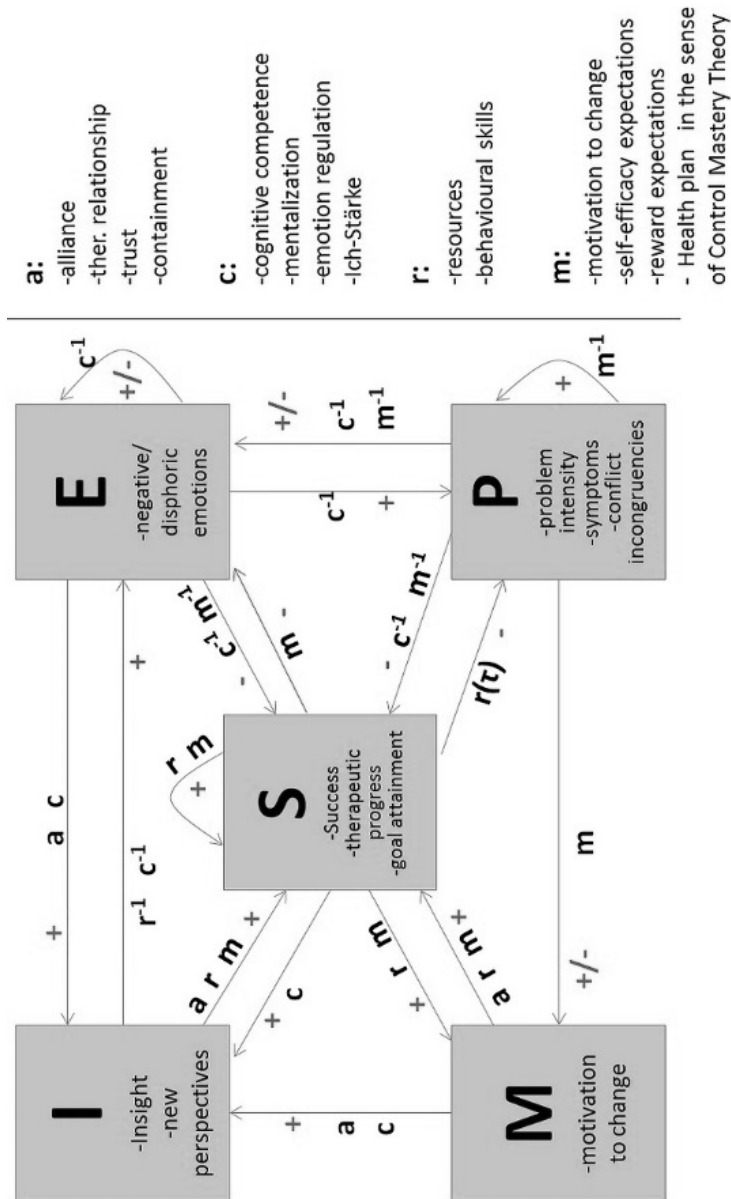
Fig. 1. The graphical system model illustrates the dependencies between the variables and the parameters of the system.

time monitoring system; Schiepek et al., 2015) and is completed by patients every day of their stay on a hospital ward or on a day treatment center (Schiepek et al., 2016).

The graphical system model (Fig. 1) illustrates the dependencies between the variables of the system (P, S, M, E, I) and the parameters mediating the functions (a, m, c, r). r^{-1}, c^{-1}, or m^{-1} indicate an inverse effect: the smaller the parameter, the bigger the effect. For example, the intensity of negative emotions reduces the experience of progress in psychotherapy, especially if cognitive competencies and self-efficacy expectations (parameters c and m) are low. In contrast, c and m protect against this effect of negative emotions on failure. In Fig. 2 the functions described in the text and in Table 1 are related to the arrows indicating the effect of one variable to another. Here also the Verhulst dynamics contributing to E are shown.
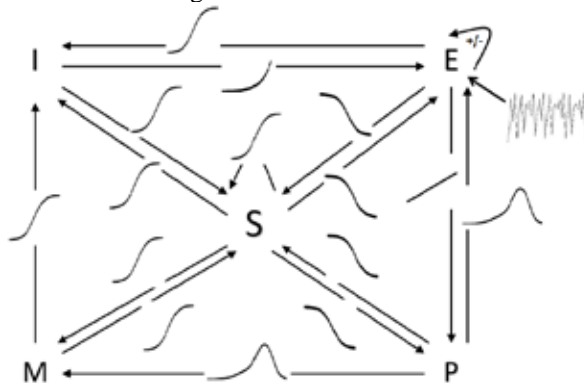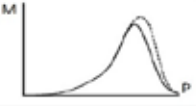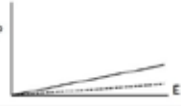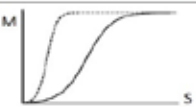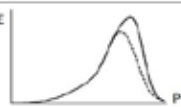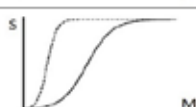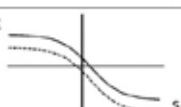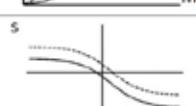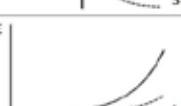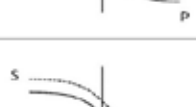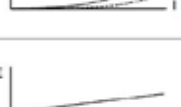

Fig. 2. The functions of the system (see text and Table 1) related to the arrows indicating the effect of one variable to an other.

The Functions

The model integrates 16 functions connecting five variables (Fig. 2). In Table 2, these functions are represented in mathematical terms (left) and designed as a graph (right). The solid and dashed graphs in the coordinate planes (x-axis: input variable, y-axis: output variable) illustrate the dependency of the shape of each function on the parameter values (solid graphs: all parameters at 0.3; dashed graphs: all parameters at 0.8).

M(P), P → M: This function describes the dependency of the actual motivation to change on the intensity of problems, conflicts, or symptom severity. It is the degree of suffering or psychological strain component of the urge to change something (avoidance goals in the sense of Grawe, 2004). If there is no problem and no suffering, there is no need to engage in problem solving. With increasing subjective problem intensity the motivation to change increases exponentially until a maximum level. Beyond this the problem seems too big to be mastered. With the problem intensity exceeding this threshold, feelings of helplessness and expectations of failure will dominate and motivation

Table 2. The 16 Functions Interrelating the Variables of the System, in Mathematical Terms (left) and Designed as a Graph (right).

$M_t(P) = \sum_{i=1}^{3} k_i m_i e^{-(P_{t-1}-b_i)^2}$ for values of k, m and b see appendix	$P_t(E) = \frac{1}{c} \cdot E_{t-1}$
$M_t(S) = \frac{1}{1+e^{-(m+r)S_{t-1}}}$	$E_t(P) = \sum_{i=1}^{3} \frac{k_i}{c_i+m_i} e^{-(P_{t-1}-b_i)^2}$ for values of k, c, m and b see appendix
$S_t(M) = \frac{1}{1+e^{-(a+m+r)M_{t-1}}}$	$E_t(S) = \frac{1}{1+e^{S_{t-1}}} - m$
$S_t(P) = \frac{1}{1+e^{P_{t-1}}} - (1-(c+m))$	$E_t(I) = (1-(c+r)) \cdot e^{I_{t-1}}$
$S_t(E) = \frac{1}{1+e^{E_{t-1}}} - (1-(c+m))$	$E_t(E) = (0.5-c) \cdot E_{t-1}$
$S_t(I) = \frac{1}{1+e^{-(a+m+r)I_{t-1}}}$	$I_t(E) = \frac{1}{1+e^{-(a+c)E_{t-1}}}$
$S_t(S) = \frac{1}{1+e^{-(m+r)S_{t-1}}}$	$I_t(S) = \frac{1}{1+e^{-cS_{t-1}}}$
$P_t(S) = \frac{1}{1+e^{S_{t-1}}} - r$	$I_t(M) = \frac{1}{1+e^{-(a+c)M_{t-1}}}$

decreases. The degree of the parameter *m* (learned self-efficacy, positive expectations in one's development, reward expectation) defines where in the range of the problem intensity this point of return is reached and how pronounced problems and psychological strain encourage the actual state of motivation to change (maximum of the function).

M(S), S → M: Success motivates. With therapeutic progress and growing confidence in a successful therapy, the motivation to engage in the therapeutic work increases. The effect of therapeutic success and reward experiences on motivation follows a logistic growth function with an inert onset

(small successful steps at first do not yet trigger big jumps in motivation) followed by an exponential increase and finally a damped effect of success on motivation. The parameters *r* and *m* determine the magnitude and steepness of the motivation gradient in the growth function. The more the client can trust his/her behavioral skills or resources, self-efficacy, and reward expectations, the more motivation will play an beneficial influential role.

S(M), M → S: Motivation supports success. With increasing motivation to engage in the therapeutic work, progress becomes more probable. Engagement is an important condition for goal attainment and accomplished steps in problem solving. Additionally, a motivation-related attention focus on self-efficacy and reward expectation is a prerequisite for progress to be perceived and esteemed. The function is a logistic growth function with an inert onset followed by an exponential increase and finally a damped effect of motivation on experienced success. The mediating parameters are *a* (quality of therapeutic alliance), *m* (reward expectation, self-efficacy), and *r* (personal resources and skills), with the assumption that these conditions help to transform motivational states into therapeutic progress.

S(P), P → S: Problem intensity has a negative impact on experienced success. If problems, symptoms, or conflicts increase (P>0), the perceived success and progress is reduced. The other way round: a decrease of problems or symptoms (P<0) will be perceived as success. The function is an inverse logistic function with the steepest effect gradient of P on S in the vicinity of P=0. Problems and symptoms have a higher negative impact on S if the parameters *c* (cognitive competencies, e.g., in self-regulation and emotion control) and *m* (reward and self-efficacy expectations) are low, and they have less reducing impact on S if *c* and *m* are high. Persons with more distinguished cognitive competencies and self-efficacy are more resilient and robust against problem exacerbations, relapses, or personal crises.

S(E), E → S: The experience of "negative" emotions like fear, grief, shame, or anger reduces (or is inversely related to) feelings of progress and being successful in solving personal problems. Within a certain range of intensity, the reducing effect on the confidence in a successful therapy course depends on the intensity of worrying emotions. This reducing effect is given by an inverse logistic function with the steepest gradient in a range of mean emotional intensity. Despite this general effect, small to middle degrees of distressing emotions can contribute to an experience of therapeutic progress (see the graph of this function in Table 1), since it can be expected that confrontation with personal conflicts, exposure to anxiety-provoking situations or imaginations, and other kinds of focusing on stressful experiences are painful but necessary as a transitional phase in personal development. "Positive" emotions (E<0) intensify the feeling of being successful and of progressing in therapy. These effects are mediated by parameters *c* and *m*, that is, by competencies in mentalization and emotion regulation, by self-efficacy, and by positive expectations in progress. The less *c* and *m* are available to a client, the more worrying emotions will reduce S.

$S(I)$, $I \rightarrow S$: Insights into the background and the psychological mechanisms of a client's problems and the development of perspectives on his/her life will create a feeling of progress in therapy. In other words, understanding is a condition for progress in problem solving, behavior change, and new qualities of interpersonal relations. The effect of I on S is mediated by a logistic growth function, with the steepness of the gradient depending on *a*, *m*, and *r*. It needs a certain degree of emotional support and safety (see the generic principle 1, Schiepek et al., 2015), given by the therapeutic relationship (*a*), as well as hopeful expectations and trust in personal development (*m*) in order to transform insight into concrete steps of behavior change (S). Of course, skills and behavioral competencies (*r*) are also necessary to transform I into S.

$S(S)$, $S \rightarrow S$: Success enhances and facilitates success. The intensity of this autocatalytic effect of S on S depends on *m* (trait motivation, self-efficacy, and reward expectation) and *r* (resources and skills).

$P(S)$, $S \rightarrow P$: Problem intensity is reduced by increasing therapeutic success and experienced progress. Positive experiences during psychotherapy (e.g., positive intra-session outcome) and steps onto a desired goal have a reducing impact on demoralization (induced by sometimes long-lasting interpersonal or personal incongruencies) or emotional problems, and therby onto the problems of a client. The effect is represented by an inverse logistic growth function with the steepest effect gradient of S on P in the vicinity of S=0. $S > 0$ reduces P, $S < 0$ increases P. The effect is mediated by *r*, that is, by the behavioral resources and skills a person can apply to the transformation of new and positive experiences made in therapeutic situations into problem solving and problem reduction in everyday situations.

$P(E)$, $E \rightarrow P$: The intensity of worrying emotions (E>0) like fear, anger, grief, or feelings of guilt contributes directly to the experience of problem intensity. In the case of affective or anxiety disorders, such emotions are by definition part of the problem or of the symptoms. The contribution of E to P is linear, with the steepness of the effect depending inversely on *c*: the smaller the capacity in emotion regulation, self-reflection, and mentalization, the more intense the contribution of E to P. However (E<0), positive feelings like joy and experiences of self-esteem reduce the problem or conflict intensity, with the steepness of this linear reducing effect also depending on *c*.

$E(P)$, $P \rightarrow E$: This function is a first approximation to a more complex relationship between P and E. It describes how an increasing problem or the intensity of a conflict activates worrying and distressing emotions. The more severe or stressing the problem, the more such emotions will be triggered. This emotion triggering effect is more pronounced if the person has only minor competencies in emotion regulation, self-reflection, and mentalization at his disposal (which are structure functions of personality in the sense of OPD) (*c*) and reduced expectations in his capacity to solve problems or to manage difficult or stressful situations (self-efficacy expectation, *m*). With higher

dispositions or competencies in *c* and *m,* coping strategies for the down-regulation of negative emotions at distinct problem intensities will be available and can be applied.

E(S), S → E: Experiences of success and therapeutic progress reduce the intensity of negative emotions and intensify positive emotions and self-esteem. This reducing effect is given by an inverse logistic function with the steepest gradient in a middle range of success. The other way round, failures or reduced therapeutic progress (S<0) intensify bad feelings. This effect is mediated by *m*, that is by self-efficacy, positive expectations in the therapeutic progress, or "trait" motivation. The stronger parameter *m* is pronounced*,* the better success and therapeutic progress will activate positive emotions and self-esteem, and the less failures or setbacks will activate worrying emotions.

E(I), I → E: In this conceptualization of psychotherapy dynamics, insight is much more based on an emotionally "hot" understanding of personally important (in-)congruencies, of conflicts, or of the impact of biographically relevant events (traumata or "life events") on someone's mental functioning than on abstract or emotionally "cold" knowledge like disease-related information in psychoeducation. This holds also true, if insight refers to new perspectives on possible scenarios of the client's life. Insight (e.g., narrative confrontation and background stories on emotionally important or even traumatic experiences) thus can activate intense emotions. The activation of emotions doesn't linearly correspond to the personal importance of the insight, but is exponentially increasing with the "intensity" or importance of the insight. The exponential function is inversely mediated by *c* and *r*: the less competencies in self-regulation or emotion regulation (*c*) and behavioral skills (*r*) are available, the more insight will trigger powerful or even uncontrollable emotions.

E(E), E → E: Depending on competencies in emotion regulation and mentalization (*c*), emotions can be up- or down-regulated. If *c* is lower than a threshold at the medium level (*c* < .5, assumed a range $0 \leq c \leq 1$), negative emotions like fear, grief, anger, or shame cannot effectively be down-regulated, and with *c* > .5, this can effectively be done. By this, *c* plays the role of a bifurcation parameter in the autocatalytic effect of E (if E>0) on itself.

I(E), E → I: As outlined in the description of the E(I) function, insight refers to an emotionally "hot" understanding of personally important topics, psychological mechanisms, conflicts, or biographically relevant events, and its impact on the client's life. In this sense, emotionally important experiences or emotion-associated states of mind (Horowitz, 1987) are a condition for such kind of "hot" insight. In terms of Grawe's general psychotherapy model, only activated negative cognitive-affective schemata can produce new qualities of understanding (Grawe, 2004) or an integration of cognitions and emotions with emerging new qualities (see the Therapeutic Cycle Model of Mergentaler, 1996). As Greenberg outlined in his emotion-focused approach, the interaction of emotion and self-related cognition (E → I) is crucial for psychotherapeutic change (Greenberg, 2002). The function I(E) is a logistic growth function with an inert onset (small intensities of stressful feelings do not yet activate negative

schemata) followed by an exponential increase and finally a damped effect of E on I. It is assumed that mid-size intensities of emotions will be optimal to create emotionally important insight. Overwhelming affects do not fulfill this effect, because they intensify self-protecting defense mechanisms and inhibit learning and self-reflection by neuronal processes (transmitter dynamics). Mediating parameters are personal competencies in self-reflection and mentalization (*c*) and the quality of the therapeutic alliance (*a*) only in a safe and appreciative interpersonal relationship intensive self-referential processes can be risked, see the "control mastery theory" (Silberschatz, 2009; Weiss, 1993).

I(S), S → I: Increases in therapeutic success or progress produce information on how problems can be solved. One aspect is the motivating effect of success (S → M) with motivation facilitating the examination of and the involvement in personal topics (M → I). Another aspect is information created by therapeutic progress. This is based on some kind of quasi-experimental relation between changed behavior (independent variable) and its effect on mental functioning, behavior, or social experiences (dependent variable) – success produces insight in the sense of information. I(S) is a logistic growth function with an inert onset followed by an exponential increase and finally a damped effect of S on I. As far as cognitive processes (information processing, mentalization, observation and reflection of one's behavior in relation to the effects on the behavior of others or oneself) are important, the parameter *c* plays a crucial role in shaping this function – its steepness depends on the value of *c*.

I(M), M → I: In order to create or construct emotionally important new insights, the client has to be motivated to do this. The attempt to establish personal meaningful relations between aspects of information may be energy consuming. So does the confrontation with conflicts or the representation of emotionally charged memories. Different states of motivation facilitate processes of self-reflection or insight by a logistic growth function, with the steepness of the gradient depending on *a* (quality of the therapeutic alliance supporting the emotionally charged process of self-reflection) and *c* (personal competencies in self-reflection and mentalization).

In addition to the dependency of emotional dynamics on experienced success, problem intensity, or insight, one can assume that emotions have their eigendynamics – corresponding to neuronal or mental "resting state" dynamics. In mental disorders we often find an intensified self-referential activity of emotions and emotion-related cognitions, such as phantasies, affect-inducing self-perceptions, affective autocatalytic processes, self-stimulation, or forced self-control (Gross, 2002; Marwaha et al., 2014; Renaud & Zaccharia, 2012). Phenomenologically this corresponds to affective instability which is defined as "rapid oscillations of intense affect, with a difficulty in regulating these oscillations on their behavioral consequences" (Marwaha et al., 2014). In depression, increased self-concern, together with deficiencies in self-regulation, result in cogitation and rumination (Northoff, 2007). In order to represent this in our model, we introduced a combined activating (positive) and inhibiting

(negative) feedback process as an autonomous eigendynamics of affects which is mathematically realized by the Verhulst map: $E'_t = kE'_{t-1}(1–E'_{t-1})$, with $0 < E' < 1$. E' indicates the eigendynamics of the emotions independent of the input from other variables (see Fig. 2). This eigendynamics of E' is an internal process of emotional (dys-)regulation, independent of external input, like stressors or therapeutic interventions; for an example see Figs. 3 and 4 in the Results section below. Together with E(P, I, S, E), E' contributes to the final dynamics of E. The driving parameter *k* is fixed at 3.8, that is, the dynamics of the Verhulst map is working in its chaotic regime (Strunk & Schiepek, 2006).

In general, the dependencies of variables on other variables of a system (or on themselves) and on the mediating parameters can be written by a set of coupled difference equations ($\mathbf{X}_t$ is the vector of the system variables, *p* is the vector of the parameters, $\mathbf{F}$ is the set of functions):

$$\mathbf{X}_t = \mathbf{F}(\mathbf{X}_{t-1}, \boldsymbol{p})$$

In this model, the functions are entering as additive terms:

$$P_t = f_1(S_{t-1}, r) + f_2(E_{t-1}, c),$$

$$S_t = f_3(P_{t-1}, c, m) + f_4(E_{t-1}, c, m) + f_5(M_{t-1}, a, m, r) +$$

$$f_6(I_{t-1}, a, m, r) + f_7(S_{t-1}, r, m),$$

$$M_t = f_8(P_{t-1}, m) + f_9(S_{t-1}, r, m),$$

$$E_t = f_{10}(P_{t-1}, c, m) + f_{11}(I_{t-1}, c, r) + f_{12}(S_{t-1}, m) + f_{13}(E_{t-1}, c) + \text{Verhulst-}$$
driven eigendynamics,

$$I_t = f_{14}(E_{t-1}, c, a) + f_{15}(M_{t-1}, a, c) + f_{16}(S_{t-1}, c),$$

All functions (see Table 1) were integrated into five coupled nonlinear difference equations (discrete maps) describing the dependencies of each variable on their input variables.

$$P_t = \frac{1}{c}E_{t-1} + \frac{1}{1+e^{S_{t-1}}} - r$$

$$S_t = \frac{1}{1+e^{-(a+m+r)I_{t-1}}} + \left(\frac{1}{1+e^{E_{t-1}}} - (1-(c+m))\right) + \left(\frac{1}{1+e^{P_{t-1}}} - (1-(c+m))\right) + \frac{1}{1+e^{-(a+m+r)M_{t-1}}} + \frac{1}{1+e^{-(m+r)S_{t-1}}}$$

$$I_t = \frac{1}{1+e^{-(a+c)E_{t-1}}} + \frac{1}{1+e^{-cS_{t-1}}} + \frac{1}{1+e^{-(a+c)M_{t-1}}}$$

$$E_t = (1-(c+r))e^{I_{t-1}} + (0.5-c)E_{t-1} + \left(\frac{1}{1+e^{S_{t-1}}} - m\right) + \sum_{i=1}^{3}\frac{k_i}{c_i+m_i}e^{-(P_{t-1}-b_i)^2} + 3.8E'_{t-1}(1-E'_{t-1})$$

$$M_t = \sum_{i=1}^{3} k_i m_i e^{-(P_{t-1}-b_i)^2} + \frac{1}{1+e^{-(m+r)S_{t-1}}}$$

The model was designed in MatLab's Simulink with a fixed step time of 1 and the 'FixedStepDiscrete' solver. Parameters were estimated using the

"Parameter Estimation Toolbox" with the implemented Levenberg-Marquardt-Algorithm.

The parameters of the model simulations were estimated from the time series data of a specific client. Additional calibration parameters for the scaling of the functions to produce effects within a range of [0, 100] or [-100, 100] are presented in the Appendix.

The presented model does not entail additional noise, that is, it runs purely deterministically. That way, one gets a better understanding of the mechanisms described in the model. With additional noise, the deterministic chaos produced by the system would be confounded by a stochastic input driving the system from the outside. In a first step it should be investigated whether the system behaves chaotically, and in next steps the effects of dynamic noise onto the dynamics could be examined. In the result section, we present simulations running over 100 or 150 iterations, since many empirical time series available in our data set portray processes within this range of measurement points (= days).

Time series are presented in z-transformed values (except in Figs. 6 and 7), since the scales of the variables are in arbitrary units and it seems more important to identify the relative patterns and dynamic qualities than the absolute values. In order to smooth the simulated time series we used a cubic spline.

RESULTS

The model generates dynamic patterns with and without the Verhulst-driven eigendynamics of E'. Figure 3 illustrates an example of two simulations based on the 16 functions described above, without the driving effect of the Verhulst dynamics. The simulation runs were realized with $a = 0.4$, $m = 0.7$, $c = 0.2$, $r = 0.1$. The simulated time series were z-transformed in order to present them in a similar scaling range (the y-axis is in units of standard deviations of the original time series) and smoothed by a cubic spline. In Fig. 3a, there is no intervention; all variables were unaffected. In Fig. 3b, the values of P_t were reduced by –60 during the iterations t = 50 to t = 60 (black bar), all other variables were unaffected. After this period, no further intervention was realized. After a transient period, the systems behavior converges to the same quasi-periodic rhythm which is realized by the dynamics without the intervention (however, in the uninfluenced dynamics this rhythm emerges earlier, Fig. 3a. Interestingly, after the end of the intervention period there is a "rebound" effect with intensified P and E, and decreased S. From about t = 100, the dynamics runs into the same dynamics as it is realized without the intervention (Fig. 3a).

Figure 4 illustrates an example of two simulations based on the 16 functions with the driving effect of the Verhulst dynamics in E' integrated. The simulations were realized with $a = 2.0$, $m = 0.1$, $c = 0.4$, $r = 0.2$. The simulated time series were z-transformed in order to present them in a similar scaling range and smoothed by a cubic spline. In Fig. 4a, there is no intervention, all variables were unaffected. During the simulation run shown in Fig. 4b, the

values of P_t were reduced by –20 during the iterations t = 50 to t = 60 (black bar); all other variables were unaffected. After this period, no further intervention was realized. As in Fig. 3, the effects of the intervention on P to S, M, and I are not enduring. After a transient period, they vanish and the system dynamics relaxes to the attractor realized before the intervention took place. The length of the relaxation period depends on parameter values (no figure).
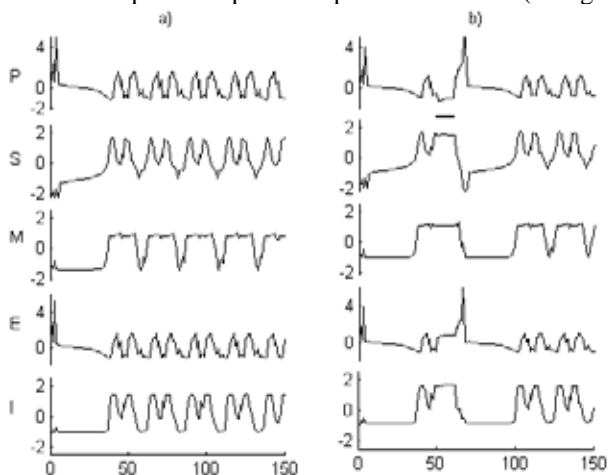
Fig. 3. Model simulations (without the driving effect of the Verhulst dynamics on E'). 3(a): Without intervention. 3(b): With intervention on P_t (black bar).
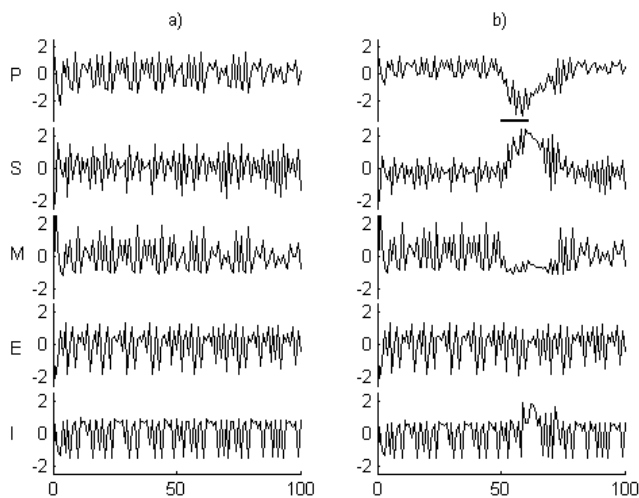

Fig. 4. Model simulations (with integrated Verhulst-driven eigendynamics on E'). 4(a): Without intervention. 4(b): With intervention on P_t (black bar).
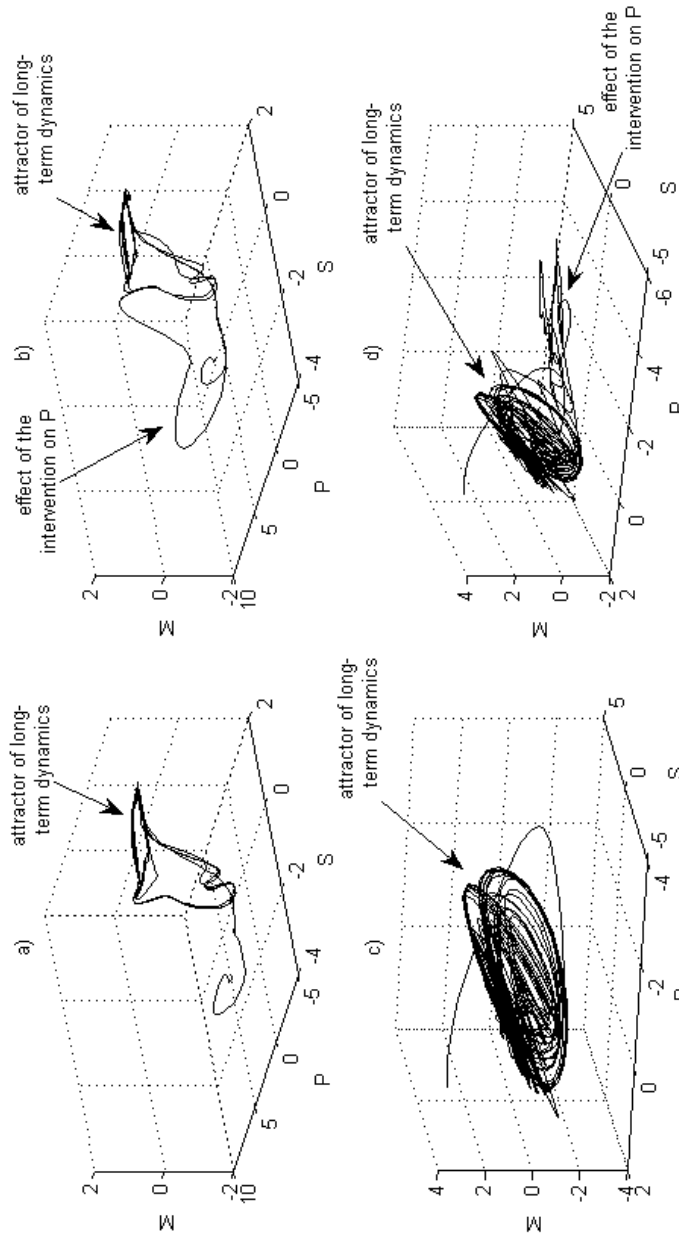
Fig. 5. Phase portraits. The simulation runs shown in Figs. 3(a, b) and 4(a, b) are embedded in corresponding phase spaces defined by the variables P, S, and M.

The time series shown in Figs. 3a, 3b, 4a, and 4b were embedded in phase spaces (Fig. 5). The variables P, S, and M define the three axes of the 3-dimensional phase spaces, i.e., each value of (P_t, S_t, M_t) defines a state vector of the system (represented by a point), and the sequence of these vectors create the trajectory of the system. I and E are not represented in Fig. 5. The trajectories are smoothed since the embedded time series of P, S, and M were smoothed by a cubic spline before the embedding; in order to make the scale ranges similar, the time series were z-transformed. The simulated time series of P, S, and M in Fig. 3a (without Verhulst, no intervention) correspond to the phase portrait in Fig. 5a, the simulated time series in Fig. 3b (with intervention) correspond to the phase portrait in Fig 5b; the time series of P, S, and M in Fig. 4a (with Verhulst, no intervention) correspond to the phase portrait in Fig. 5c, the simulated time series in Fig. 4b (with intervention) correspond to the phase portrait in Fig. 5d. In both cases, the attractors of the unaffected dynamics were re-established after the intervention. The "out of attractor" dynamics during the period of intervention is indicated by arrows in Fig. 5.

Without therapeutic manipulations and without any changes of the trait parameters *a*, *c*, *r*, or *m,* the system produces stable dynamics. Synergetics and other theories of complex systems predict that phase transitions depend on changed control parameters. For new attractors to emerge, a single intervention in the dynamics of variables (e.g., a short-term reduction of problems) is not sufficient. It takes a change of parameters in order to receive a persistent new behavior (attractor). The effect of our modelled intervention is a short-term reduction in P affecting S, M, and E (Fig. 3b, or S, M, and I in Fig. 4b), but after a transition period, the dynamics relaxes to the original attractor realized without intervention (Fig. 5).

The dynamic patterns of psychotherapeutic processes depend sensitively on the parameter values. This can be illustrated by comparing a virtual client with higher resources to a virtual client with lower resources (resources here are defined by the client's attachment and collaboration competencies, parameter *a;* self-efficacy expectations, *m*; cognitive and mentalization competencies, *c*; behavioral skills, *r*). In the virtual low resources client, we expect higher levels and – due to reduced competencies in affect regulation – also more pronounced fluctuations in problem intensity (P), as well as a lower level of experienced success (S). Reduced success and achievement experiences would reduce the motivation to change (M), while more problems might trigger the urge to change (strain and degree of suffering component of M). In consequence M should be more pronounced and more shifting between upward (urge to change) and downward (resignation) peaks. E (the level of negative emotions) should be lower in the "high resources" case; for I, we do not have specific expectations.

In the simulation shown in Fig. 6a, the virtual client has lower competencies in all parameters ($a = 0.2$, $m = 0.2$, $c = 0.2$, $r = 0.2$) than the client in the simulation Fig. 6b ($a = 0.7$, $m = 0.4$, $c = 0.7$, $r = 0.3$). In order to get a direct comparison of the levels and dynamic patterns of the simulations, the time series were re-scaled onto ranges from [0, 1] (for P, M, I), [-1, 1] (for E), or [-1, 0] (for S) and smoothed by a cubic spline. The simulation is based on the model with the Verhulst dynamics in E'. As expected for the low resources client the dynamics of P evolves at a higher level with more pronounced fluctuations (Fig.
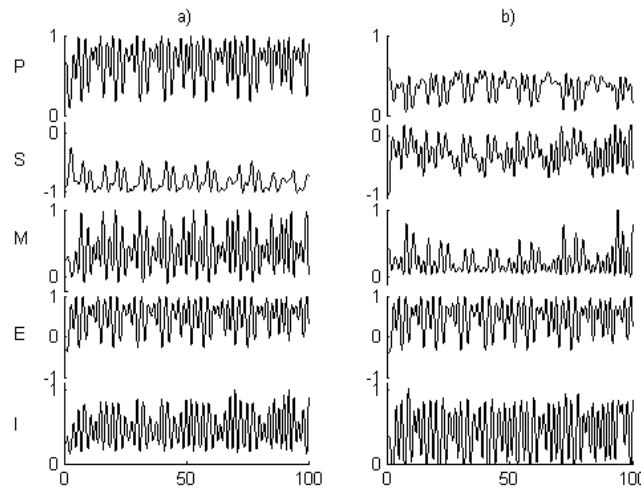
Fig. 6. An illustration of the effects of parameter values on the system dynamics. A comparison between the simulation runs of a virtual "low resources client" (6a) and a virtual "high resources client" (6b).

6a), and the dynamics of S evolves at a lower level than in the high resources client (Fig. 6b). M of the virtual client with low resources, moves at a higher level and with more upward and downward fluctuations than the M of the high resources client (Fig. 6b). Evidently, lower degrees of problem intensity implicate less motivation to change. This corresponds to the predictions. Differences in E or I between the two cases are only marginal.

The sensitive dependency of the dynamics to the parameter values is illustrated by parameter-dependent bifurcation diagrams (Fig. 7). (Here we refer to the model without the Verhulst-driven eigendynamics of E'.) These diagrams show the values of the dynamics at a certain parameter adjustment, realized within a time window of 1000 iterations beyond a transition period of 200 iterations. The transition period is eliminated and the following 1000 values are used to characterize the realized values of each parameter-specific attractor – a procedure which is known from the famous Feigenbaum scenario of the Verhulst map (see Strunk & Schiepek, 2006). In the parameter range of [0, 1] *a*, *c*, *r*, and *m* are discriminated by steps of 0.001. We thus depict 1000 segments, with each segment displaying 1000 values of the respective variable plotted in a sliced vertical line. The adjustment of the parameters that do not vary in the respective diagram is fixed at 0.5. In order to get similar attractor ranges the simulated time series of each variable were rescaled by a certain divisor (P by 5, S by 3, M by 1.2, E by 15, I by 2.5). Figure 7 illustrates the varying complexity of the attractors (between fix points, different kinds of periodicities, and a great diversity of chaotic dynamics) and thereby the parameter-dependent multistability of the model.
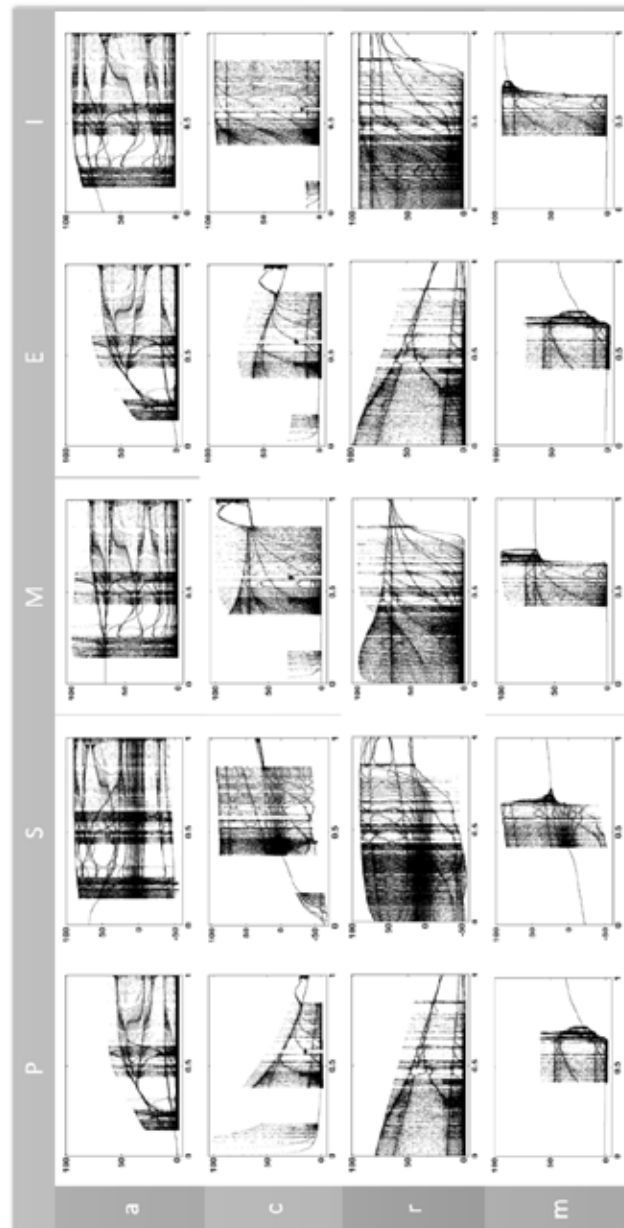
Fig. 7. Parameter-dependent bifurcation diagrams. Parameter ranges [0,1] are subdivded by steps of 0.001. The non-varying parameters are fixed at 0.5, respectively.

The similarity of the simulated dynamics to an empirically assessed psychotherapy process is illustrated in Fig. 8. In Fig. 8b, the dynamics of S and P is taken from the empirical time series of a real client (diagnosed as Major Depressive Disorder) with S and P corresponding to the factors "therapeutic progress and confidence in a successful development" and "complaints and problem pressure of the Therapeutic Process Questionnaire (Haken & Schiepek, 2010). The parameters of the simulation (Fig. 8a) were estimated by the empirical time series data of the client with $a = 0.5$, $m = 0.1$, $c = 0.4$, $r = 0.2$, and the initial conditions corresponding to its initial values. In the simulation, an intervention was applied to P_t which was stepwise reduced by –10 (from t = 45 to 49), –12.5 (from t = 50 to 54), –15 (from t = 55 to 59), –17.5 (from t = 60 to 69), and –20 (from t = 65 to 100). The simulated time series were z-transformed and smoothed by a cubic spline. A more rigorous and systematic comparison of simulated and empirical dynamics (e.g., by using initial values and parameters from real clients) is work in progress and will be presented in a separate paper.
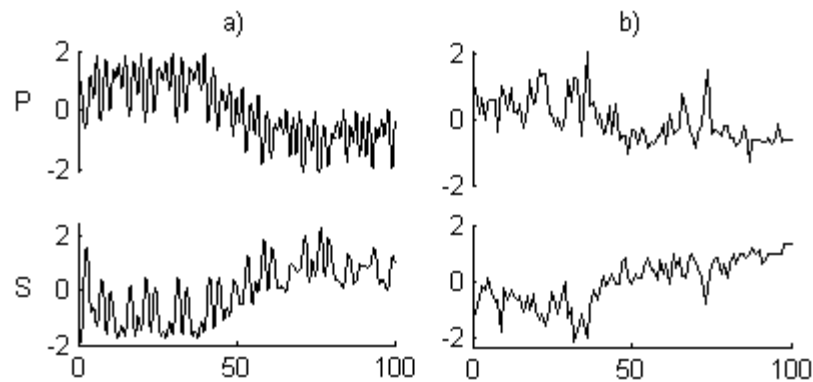

Fig. 8. Dynamics of the variables P and S from a model simulation (8a) and from a real client (8b).

DISCUSSION

The aim of this study was the development of a mathematical formalism which is able to reproduce therapeutic change processes by means of a computer simulation. The set of equations proposed generates a great variety of dynamics and demonstrates that a model-based simulation of psychotherapeutic change processes is possible. The main focus of this contribution was on the qualitative model behind its mathematical formalism, which is given by five coupled nonlinear difference equations. Each of the equations describes the interdependence of the variables P (problem or symptom severity), S (success and therapeutic progress), M (motivation to change), E (intensity of worrying or positive emotions), and I (insight and new perspectives on problems) by 14 nonlinear and two linear functions. These functions are represented by the 16 additive terms of the equations. The shape of each

function is modified by four parameters (*a*: capability to form a trustful attachment and working alliance, *c*: cognitive competencies in mentalization and emotion regulation, *r*: behavioral resources and skills, *m*: self-efficacy and reward expectation). In an extended model, E is driven by an intrinsic eigendynamics produced by the Verhulst map in its chaotic regime. The variables and parameters could be further differentiated and the model expanded. However we aimed at reaching maximum explanatory power (i.e., to cover the most important mechanisms and to produce a wide variety of dynamic phenomena) by a slim and yet powerful model. The criterion is not complexity or complexity reduction at any price, but "adequate" complexity.

The model demonstrates that the patterns of change depend on the parameter values which are – in psychological terms – the dispositions or competencies of a client entering the challenging process of psychotherapy and its interwoven cognitive, affective, behavioral, and interpersonal subprocesses. The time-limited manipulation of a variable (e.g., symptoms) is not sufficient to change the existing problem attractor. After the intervention, the dynamics relaxes to the formerly existing attractor – sooner or later, with or without rebound effects. Both models (with and without the Verhulst-driven eigendynamics on E') behave chaotically, depending on the specific values and ranges of parameters. The models as presented in this paper are completely deterministic and noiseless. The most important result is the mere fact that a great deal of information on psychological mechanisms and on common factors contributing to psychotherapy effects can be compressed in a formalism of five equations which generates a great diversity of dynamics.

The limitations of this model of change dynamics are in the parametrization of the equations and in the definition of the psychological model. In the specification of the model we were forced to introduce some additional parameters to calibrate the dynamics in a stable range. In an additional optimization of the model, these theoretically undefined calibration parameters should be eliminated. Given the variety of possible common factors discussed in the literature (see the remarks in the Introduction paragraph) there is room in selecting and defining the factors that enter the model. As stated in the "Model" paragraph, only M and I are common factors in a strict sense, P and S are micro-outcomes. Client factors such as resources, cognitive competencies, or self-efficacy and reward expectations are represented by the parameters. Since the model tries to operate with the smallest possible number of elements, the variables and parameters used are somewhat fuzzy and cover a broad range of meaning. For example: self-efficacy is part of *m* but concrete experiences of self-efficacy are based on changes in M and S, with M and S contributing to increased self-efficacy expectations. Other categories discussed in the common factors literature are features of the dynamics and in consequence are not specified as variables or parameters of their own, but result from their interaction. The experienced therapeutic relationship or working alliance enters at the attachment or alliance parameter *a*. Instead of a fixed value, the development trajectory of the client's alliance (as represented by the therapeutic

relationship subscale of the TPQ) could be used as an input vector of the parameter a. Also, therapist variables are not explicitly represented in the model but result from the reactiveness to the client, entering the model by the interventions and by the resonance to the client's system dynamics.

With our paper, we present a small first step towards a formalized understanding of psychotherapy. Ultimately, this endavour might form a sound theoretical ground for a systems psychotherapy. On this way we have to ask – as Haken (1992) did in the mathematization of perception and movement – "How far can we go?" In further steps, we believe that the 4-dimensional parameter space could be analyzed to get a concrete portrait of the parameter ranges producing stable dynamics. The hypothesis is that specific parameter values correspond to specific disorders, producing disease-specific dynamic patterns. Another step is to advance the details and specify some functions of the model, first of all E(P). It should be able to represent different affective reaction patterns stimulated by different degrees of problem intensities.

A third step is to understand how psychotherapy creates long-term effects. In consequence, the effects of state dynamics on trait dynamics have to be introduced into the model (up to now, this is only done for the effects of traits on states). Technically, the change of parameters as a function of the variables has to be implemented by a set of equations responsible for the slower parameter shift depending on the faster dynamics of variables and on the input from the outside (e.g., treatment-specific input, social interactions, noise from daily hassles). Parameter dynamics and variable dynamics evolve at different time scales (time scale differentiation) which helps to understand how short-term variations in experiences, emotions, and behavior will create changes in dispositions. In other words: How do states influence traits?

A systematic model testing should be realized by simulated experiments. In these experiments, a systematic variation of inputs onto the variables could create a complete portrait of the model's behavior spectrum and its multi-stability. Model testing includes also a comparison of simulated dynamics with the empirical dynamics. In order to do so, we gathered empirical data from 753 persons of different diagnoses that received psychotherapeutical treatment.

Nonlinear models like this raise doubt on the meaningfulness of the percent-values of explained variance accredited to common or specific factors. Within the paradigm of complexity science, factors do not contribute to the outcome in an additive way, but are nonlinearly interacting in a network producing self-organized and chaotic behavior where small fluctuations can have important effects and powerful interventions might have no long-term impact onto the outcome at all.

In the future, computer simulations will also provide tools for the training of psychotherapists in understanding complex, self-organizing systems and acting in real life scenarios where contraintuitive behavior and limited

prediction horizons is the norm. Psychotherapists should become competent experts in understanding and modeling complex bio-psycho-social systems.

APPENDIX:
CALIBRATION OF THE FUNCTIONS AND INTERCONNECTIONS AMONG VARIABLES

The calibration of the functions of the system is given by the following values. The calibration should limit the simulated time series onto a range of [0, +100] or [-100, +100].

```
E(E)=2*(0.5-c)*E

E(I)=100*(1-((c+r)/2))*exp(0.05*(I-100))

E(P)=4*(2/(c+m))^2*exp(-((P-82)/8)^2)+57*exp(-((P-
      72)/12)^2)+25*exp(-((P-60)/25)^2)

E(S)=125/(1+exp(0.05*S-0.5))-50-25*(2*m)

I(E)=100/(1+exp(-E*0.2*((a+c)/2)+5))

I(M)=100/(1+exp(-M*0.2*((a+c)/2)+5))

I(S)=100/(1+exp(-S*0.2*c+5))

M(P)=50*m^2*exp(-((P-82)/8)^2)+57*exp(-((P-
      72)/12)^2)+25*exp(-((P-60)/ 25)^2)

M(S)=100/(1+exp(-S*0.2*((m+r)/2)+5))

P(E)=E/(10*c)

P(S)=150/(1+exp(0.05*S-0.5))-68-25*(2*r)

S(E)=115/(1+exp(0.05*E-0.5))-50-25*(1-(c+m))

S(I)=100/(1+exp(-I.*0.2*((a+m+r)/3)+5))

S(M)=100/(1+exp(-M.*0.2*((a+m+r)/3)+5))

S(P)=100/(1+exp(0.05*P-0.5))-62-25*(1-(c+m))

S(S)=100/(1+exp(-S*0.2*((m+r)/2)+5))

Verhulst=200*(3.8*x*(1-x))-75
```

An additional coupling stregth of the functions interconnecting the variables is estimated from the empirical time series of a specific client in Table 3. P and S of this case are shown in Fig. 8b). Columns (a) in Table 3 contains coupling strength parameters used in the model with Verhulst-driven E. Columns (b) contains coupling strength parameters used in the model without Verhulst-driven E.

Table 3. Coupling Strength Parameters (a) With and (b) Without Verhulst-driven E.

	a)	*b)*		*a)*	*b)*
E(E)	0.00	5 E-5	I(E)	0.40	0.26
E(I)	0.11	0.02	I(M)	1.00	0.64
E(P)	0.08	3.99	I(S)	2.29	1.46
E(S)	0.06	0.38	M(P)	0.57	0.47
	a)	b)		a)	b)
M(S)	0.90	0.82	S(I)	0.49	0.45
P(E)	1.40	1.34	S(M)	1.00	0.92
P(S)	0.34	2 E-3	S(P)	1.07	2.44
S(E)	0.42	0.41	S(S)	1	0.92

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